Free vibration of single and sandwich laminated composite plates by using a simplified FSDT

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Abstract.

This paper presents a simplified first order shear deformation theory (FSDT) for laminated composite and sandwich plates. Unlike the existing FSDT, the present one has a novel displacement field which include undetermined integral terms and contains only four unknowns. Equations of motion and boundary conditions are derived from the Hamilton's principle. Navier-type analytical solution is obtained in closed form and by solving the eigenvalue equation. The comparison of the present results with the available elasticity solutions and the results computed independently using the FSDTs available in the literature shows that this theory predicts the fundamental frequencies with good accurately. It can be concluded that the proposed theory is accurate and simple in solving the dynamic behaviour of single and sandwich laminated composite plates.

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1. Introduction

Laminated composite materials are extensively used in aerospace, marine, civil and other areas. With their high specific modulus, high specific strength, and the capability of being tailored for a specific application, laminate composites offer definite advantages compared to traditional materials like for example metal. The individual layer consists of high-modulus, high-strength fibers in a polymeric, metallic, or ceramic matrix material. With the ongoing development of the high-tech industry, demand for advanced materials has led to the development of substitutes for traditional engineering materials such as wood, aluminum, steel, concrete, etc. Consequently, new methodology to study the behaviour of such materials is still desirable. Among the recent refined mathematical models to study the bending, vibration, buckling, etc., several classical theories to study the laminated composite plates were previously developed.

The classical laminate plate theory (CLPT), which neglects the transverse shear effects, provides reasonable results for thin plates. However, underpredicts deflections and over predicts frequencies as well as buckling loads with moderately thick plates. Many shear deformation theories that account with transverse shear effects have been developed to overcome such problem. Consequently, as improvement the FSDT and HSDT (High shear deformation theory) were developed. The classical FSDT is based on Reissner [1] and Mindlin [2] and account for the transverse shear effects by linearly modelling the displacements through the thickness. Since FSDTs violates equilibrium conditions at the top and bottom faces of the plate, shear correction factors are required to rectify the unrealistic variation of the shear strain/stress through the thickness.

Many studies have been carried out using FSDT for the free vibration analysis of composite plates [3-13]. Kant [11] reproduced the FSDT given by Whitney and Pagano [5] for the free vibration analysis of laminated composite and sandwich plates. Thai [12] also reproduced the FSDT given by Whitney and Pagano [5] but only for the free vibration analysis of laminated composite plate. Thai [13] propose a new FSDT and obtained values for fundamental frequency of antisymmetric cross-ply square laminates.

In order to overcome the limitations of FSDT, polynomial HSDTs which involve higher-order terms in Taylor’s expansions of the displacements in the thickness coordinate, were developed by Librescu [14], Levinson [15], Bhimaraddi and Stevens [16], Reddy [17], Ren [18], Kant and Pandya [19], and Mohan et al. [20]. A good review of these theories for the analysis of laminated composite plates is available in [21–25]. In what follows some papers are consider in this short survey.

A two variable refined plate theory was developed by Shimpi [26] for isotropic plates, and extended by Shimpi and Patel [27,28] for orthotropic plates. Reddy [29] developed a polynomial HSDT with cubic variations for in-plane displacements. Xiang et al. [30] proposed a n-order shear deformation theory in which Reddy’s theory comes out as special case. Kant and Pandya [31], Kant and Mallikarjuna [32] and Kant and Khare

[33] presented also polynomial HSDTs with cubic variations for in-plane displacements as in the paper by Reddy [29]. To account for the thickness stretching effect (i.e., $\varepsilon_z \neq 0$), Lo et al. [35] and Kant et al. [36] introduced HSDTs in which in-plane and transverse displacements are assumed as cubic and parabolic variations through the thickness, respectively. In addition to the aforementioned works, more recently, Mantari et al. [37–39] proposed the use of several trigonometric functions for in-plane and transverse displacement field.

It is worth noting that some of the abovementioned HSDTs are computational costly due to additional unknowns introduced to the theory (e.g., theories by Refs. [31,32] with seven unknowns, Ref. [33] with nine unknowns, Refs. [35,36] with 11 unknowns). Moreover, in many of the abovementioned HSDTs as in the CPT or the simple FSDT proposed by Thai [13], the expression $\partial w/\partial x$ or $\partial\theta/\partial x$ are present in the displacement field. Consequently, the numerical computation is harder to handle. Normally C^1-FEM is required. However, this can be changed if the displacement field is composed with undetermined integral terms as in this paper.

In this paper, a simplified FSDT for the dynamic study of single and sandwich laminated composite plates is presented. The addition of the integral term in the displacement field leads to a reduction in the number of unknowns and governing equations. Closed-form solutions of simply supported antisymmetric cross-ply laminates are obtained. The results are in many cases equal to the classical FSDT. Consequently, the capability of the present FSDT is validated.

2. Theoretical Formulation

2.1. Kinematics:

In this study, further simplifying assumptions are made to the existing FSDT so that the number of unknowns is reduced. The displacement field of the existing FSDT is given by Whitney and Pagano [99]:

$$\begin{aligned}
\bar{u}(x,y,z) &= u(x,y) + z\theta_x, \\
\bar{v}(x,y,z) &= v(x,y) + z\theta_y, \qquad \text{(1a-c)} \\
\bar{w}(x,y,z) &= w(x,y).
\end{aligned}$$

where $u(x,y)$, $v(x,y)$, $w(x,y)$, $\theta_x(x,y)$ and $\theta_y(x,y)$ are five unknown displacement functions of the mid-plane of the plate. In this paper a novel displacement field with 4 unknowns is proposed:

$$\begin{aligned}
\bar{u}(x,y,z) &= u(x,y) - k_1 z \int \theta dx, \\
\bar{v}(x,y,z) &= v(x,y) - k_2 z \int \theta dy, \qquad \text{(2a-c)} \\
\bar{w}(x,y,z) &= w(x,y).
\end{aligned}$$

where $u(x,y)$, $v(x,y)$, $w(x,y)$ and $\theta(x,y)$ are the four unknown displacement functions of middle surface of the panel. The constants k_1 and k_2 depends on the geometry. The integrals used are undetermined. In the derivation of the necessary equations, small strains are assumed (i.e., displacements and rotations are small, and obey Hooke's law). The linear strain expressions derived from the displacement model of Equations (2a-c), valid for thin, moderately thick and thick plate under consideration are as follows:

$$\begin{aligned}
\varepsilon_{xx} &= \varepsilon_{xx}^0 + z\varepsilon_{xx}^1 \\
\varepsilon_{yy} &= \varepsilon_{yy}^0 + z\varepsilon_{yy}^1 \\
\gamma_{xy} &= \gamma_{xy}^0 + z\gamma_{xy}^1 \qquad \text{(3a-e)} \\
\gamma_{xz} &= \gamma_{xz}^0 \\
\gamma_{yz} &= \gamma_{yz}^0
\end{aligned}$$

where

$$\begin{aligned}
\varepsilon_{xx}^0 &= \frac{\partial u}{\partial x}, & \varepsilon_{xx}^1 &= -k_1\theta, \\
\varepsilon_{yy}^0 &= \frac{\partial v}{\partial y}, & \varepsilon_{yy}^1 &= -k_2\theta, \\
\gamma_{xy}^0 &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, & \gamma_{xy}^1 &= -k_1\frac{d}{dy}\int \theta dx - k_2\frac{d}{dx}\int \theta dy, \qquad \text{(4a-h)} \\
\gamma_{xz}^0 &= -k_1\int \theta dx + \frac{\partial w}{\partial x}, \\
\gamma_{yz}^0 &= -k_2\int \theta dy + \frac{\partial w}{\partial y},
\end{aligned}$$

2.2 Constitutive equations:

The linear constitutive relations are given below:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}. \tag{5}$$

in which, σ={ σ_{xx}, σ_{yy}, τ_{xy}, τ_{xz}, τ_{yz} }^{T} and ε={ ε_{xx}, ε_{yy}, γ_{xy}, γ_{xz}, γ_{yz} }^{T} are the stresses and the strain vectors with respect to the plate coordinate system. Where Q_{ij} are the material constants in the material axes of the layer given as:

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}},\ Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}},\ Q_{12} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}},\ Q_{66} = G_{12},\ Q_{55} = G_{13},\ Q_{44} = G_{23}$$

where E is the modulus of elasticity in the main direction, G is the shear modulus in the plane.

2.3. *Hamilton's Principle:*

Hamilton's Principle is applied to the present case, the following expressions can be obtained:

$$0 = \delta \int_{t1}^{t2} [K - U] dt \tag{6}$$

where U is the total strain energy due to deformations and K the kinetic energy.
Substituting the appropriate energy expressions can be obtained:

$$0 = -\int_{t1}^{t2} \left[\int_{-h/2}^{h/2} \left\{ \int_{A}^{(k)} \left[\sigma_{xx}\delta\varepsilon_{xx}^{(k)} + \sigma_{yy}\delta\varepsilon_{yy}^{(k)} + \sigma_{xy}\gamma\varepsilon_{xy}^{(k)} + \sigma_{yz}\gamma\varepsilon_{yz}^{(k)} + \sigma_{xz}\gamma\varepsilon_{xz}^{(k)} \right] dxdy \right\} dz \right] dt$$

$$+ \int_{t1}^{t2} \left[\int_{-h/2}^{h/2} \int_{\Omega} \rho \left(\dot{\bar{u}}\, \delta \dot{\bar{u}} + \dot{\bar{v}}\, \delta \dot{\bar{v}} + \dot{\bar{w}}\, \delta \dot{\bar{w}} \right) dV \right] dt, \tag{7}$$

substituting corresponding terms,

$$0=\int_{t1}^{t2}\left\{\int_{\Omega}\left[-(N_1\delta\varepsilon_{xx}^0+M_1\delta\varepsilon_{xx}^1+N_2\delta\varepsilon_{yy}^0+M_2\delta\varepsilon_{yy}^1+N_6\delta\gamma_{xy}^0+M_6\gamma\varepsilon_{xy}^1+Q_1\gamma\varepsilon_{yz}^0+Q_2\gamma\varepsilon_{xz}^0)+\right.\right.$$

$$+((-I_1\ddot{u}-k_1\frac{I_2}{\alpha^2}\frac{\partial\ddot{\theta}}{\partial x})\delta u+(-I_1\ddot{v}-k_2\frac{I_2}{\beta^2}\frac{\partial\ddot{\theta}}{\partial y})\delta v+(-I_1\ddot{w})\delta w+(k_1\frac{I_2}{\alpha^2}\frac{\partial\ddot{u}}{\partial x}+k_1^2\frac{I_3}{\alpha^4}\frac{\partial^2\ddot{\theta}}{\partial x^2}+$$

$$\left.\left.k_2\frac{I_2}{\beta^2}\frac{\partial\ddot{v}}{\partial y}+k_2^2\frac{I_3}{\beta^4}\frac{\partial^2\ddot{\theta}}{\partial y^2})\delta\theta)\right]dxdy\right\}dt \tag{8}$$

where $\varepsilon^{(k)}$ or $\sigma^{(k)}$ are the stresses and the strain vectors, α and β are derived from Navier method (13), and N_i, M_i, and Q_i are the resultants of the following integrations:

$$(N_i, M_i)=\sum_{k=1}^{N}\int_{z^{(k-1)}}^{z^{(k)}}\sigma_i^{(k)}(z)(1,z)dz, \qquad (i=1, 2, 6)$$

$$(Q_1)=\sum_{k=1}^{N}\int_{z^{(k-1)}}^{z^{(k)}}K\sigma_4^{(k)}(z)dz,$$

$$(Q_2)=\sum_{k=1}^{N}\int_{z^{(k-1)}}^{z^{(k)}}K\sigma_5^{(k)}(z)dz. \tag{9a-c}$$

where K is the shear correction factor. The inertia term are defined by the following integrations:

$$(I_1, I_2, I_3)=\int_{-h/2}^{+h/2}\rho(1,z,z^2)dz \tag{10}$$

2.4. Plate governing equations

Using the generalized displacement–strain relations (Equations (3a-e) and (4a-h)) and stress–strain relations (Equation 5), and applying integrating by parts and the fundamental lemma of variational calculus and collecting the coefficients of δu, δv, δw y $\delta\theta$ in Equation 8, the equations of motion are obtained as:

$$\delta u: \quad \frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} = I_1 \ddot{u} + k_1 \frac{I_2}{\alpha^2} \frac{\partial \ddot{\theta}}{\partial x},$$

$$\delta v: \quad \frac{\partial N_2}{\partial y} + \frac{\partial N_6}{\partial x} = I_1 \ddot{v} + k_2 \frac{I_2}{\beta^2} \frac{\partial \ddot{\theta}}{\partial y},$$

$$\delta w: \quad \frac{\partial Q_2}{\partial x} + \frac{\partial Q_1}{\partial y} = I_1 \ddot{w},$$

$\delta\theta$:

$$k_1 M_1 + k_2 M_2 - \left(\frac{k_1}{\alpha^2} + \frac{k_2}{\beta^2}\right)\frac{\partial^2 M_6}{\partial x \partial y} + \frac{k_1}{\alpha^2}\frac{\partial Q_2}{\partial x} + \frac{k_2}{\beta^2}\frac{\partial Q_1}{\partial y} = -k_1 \frac{I_2}{\alpha^2}\frac{\partial \ddot{u}}{\partial x} - k_1^2 \frac{I_3}{\alpha^4}\frac{\partial^2 \ddot{\theta}}{\partial x^2} - k_2 \frac{I_2}{\beta^2}\frac{\partial \ddot{v}}{\partial y} - k_2^2 \frac{I_3}{\beta^4}\frac{\partial^2 \ddot{\theta}}{\partial y^2}$$

(11a-d)

Note that by the Navier method, the following equality can be obtained: $Q_1 \delta\left(\int \theta dy\right) = \frac{1}{\beta^2}\frac{dQ_1}{dy}\delta\theta,\; Q_2 \delta\left(\int \theta dx\right) = \frac{1}{\alpha^2}\frac{dQ_2}{dx}\delta\theta$, where: $k_1 = \alpha^2$, $k_2 = \beta^2$.

In what follows, the problem under consideration is solved for the simply supported boundary conditions and they are given at all four edges as follows:

$$\begin{aligned}
&u(x,0) = u(x,b) = v(0,y) = v(a,y) = 0,\\
&w(x,0) = w(x,b) = w(0,y) = w(a,y) = 0,\\
&N_2(x,0) = N_2(x,b) = N_1(0,y) = N_1(a,y) = 0,\\
&M_2(x,0) = M_2(x,b) = M_1(0,y) = M_1(a,y) = 0,\\
&\theta\,(x,0) = \theta\,(x,b) = \theta\,(0,y) = \theta\,(a,y) = 0.
\end{aligned} \tag{12a-e}$$

3. Analytical Solution

For the analytical solution of the partial differential equations (11a-d), the Navier method, based on double Fourier series, is used under the specified boundary conditions (Equations 12a-e). Using Navier's procedure, the solution of the displacement variables satisfying the above boundary conditions can be expressed in the following Fourier series:

$$u(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn} \cos(\alpha x)\sin(\beta y), \qquad 0 \le x \le a;\; 0 \le y \le b \tag{13a}$$

$$v(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn} \sin(\alpha x)\cos(\beta y), \qquad 0 \le x \le a;\; 0 \le y \le b \tag{13b}$$

$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}\sin(\alpha x)\sin(\beta y), \qquad 0\le x\le a;\ 0\le y\le b \tag{13c}$$

$$\theta\,(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Theta^{1}_{mn}\sin(\alpha x)\sin(\beta y), \qquad 0\le x\le a;\ 0\le y\le b \tag{13d}$$

where

$$\alpha=\frac{m\pi}{a},\ \beta=\frac{n\pi}{b}. \tag{14}$$

Substituting Solution functions (13a-d) into Equations (11a-d), the following equations are obtained,

$$([K]-\omega^2[M])\{\Delta\}=\{0\} \tag{15}$$

where [K] and [M] is stiffness and mass matrices respectively and [Δ] is the column vector of coefficients $\left[U_{mn},V_{mn},W_{mn},\Theta_{mn}\right]^T$.

The natural frequencies can be found from the nontrivial solution of Equation (15).

4. Numerical results and discussions

In this section, various numerical examples are described and discussed to verify the accuracy of the present theory. For verification purpose, the obtained results are compared with the exact 3D solution and those predicted by other plate theories. The description of various plate theories and their corresponding number of unknowns are listed in Table 1. In addition, the results of the classical FSDT are also calculated independently in this study. In all examples, a shear correction factor of 5/6 is used. The following lamina properties are used [40] (Material 1):

$E_1/E_2=open$, $G_{12}=G_{13}=0.6E_2$, $G_{23}=0.5E_2$, $\nu_{12}=0.25$

For convenience, the following dimensionless form is used:

- Example 1 and 2: $\overline{\omega}=\dfrac{\omega b^2}{h}\sqrt{\dfrac{\rho}{E_2}}$
- Example 3: $\overline{\omega}=\dfrac{\omega b^2}{h}\sqrt{\left(\dfrac{\rho}{E_2}\right)_f}$

Example 1. A thick antisymmetric cross-ply $(0/90)_n$ square laminate with $a/h=5$ is analyzed using Material 1. Dimensionless fundamental frequencies are given in Table 2 for various values of modulus ratio and ply number. The obtained results are compared with the exact 3D solutions reported by Noor [40] and those generated by Thai [12-13] using FSDT.

Here also the results obtained by the present FSDT are almost identical with those predicted by existing FSDT [12] and FSDT [13]. This statement is also firmly demonstrated in Fig. 2 in which the results obtained by the present theory and FSDT [12-13] are in excellent agreement for a wide range of values of modulus ratio. Readers may also see the obtained results by Kant [11], these values are different from those obtained by FSDT [12-13] (reproduced strategically besides the presentation of this paper) and the present FSDT. The FSDT results by Kant [11] need to be carefully revised.

Example 2. A simply supported antisymmetric cross-ply laminates with the thickness ratio varied from 2 to 100 and number of layers varied from 2 to 10 is analyzed using Material 1 ($E_1/E_2 = 40$). The natural frequencies computed using various models for two, four, six and ten layer antisymmetric cross-ply square laminates are given in Table 3. The obtained results are compared with the TSDT developed by Reddy [34] and those generated by Thai [12] using FSDT. Here also the results obtained by the present FSDT are almost identical with those predicted by existing FSDT [12]. This statement is also firmly demonstrated in Fig. 3 in which the results obtained by the present theory and FSDT [12] shows excellent agreement of natural frequencies of two layer $(0/90)$ and six layer $(0/90)_3$ square laminates for various thickness ratios.

Example 3. The variation of fundamental frequency with respect to the various parameters like the side to thickness ratio (a/h), thickness of the core to thickness of flange (t_c/t_f) and the aspect ratio (a/b) of a five layer sandwich plate with antisymmetric cross-ply face sheets using all the models are given in tabular form in Table 4-6. The following of material properties are used for the face sheets and the core [34], (Material 2):

Face sheets (Graphite-Epoxy T300/934)

$$E_1 = 19x10^6\,psi(131GPa),\; E_2 = 1.5x10^6\,psi(10.34GPa), E_2 = E_{3,}$$

$$G_{12} = 1x10^6\,psi(6.895GPa),\; G_{13} = 0.90x10^6\,psi(6.205GPa),$$

$$G_{23} = 1x10^6\,psi(6.895GPa), \nu_{12} = 0.22, \nu_{13} = 0.22, \nu_{23} = 0.49,$$

$$\rho = 0.057lb/inch^3(1627kg/m^3)$$

Core properties (Isotropic)

$$E_1 = E_2 = E_3 = 2G = 1000\,psi(6.89x10^{-3}GPa),$$

$$G_{12} = G_{13} = G_{23} = 500\,psi(3.45x10^{-3}GPa),$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.3, \rho = 0.3403x10^{-2}\,lb/inch^3(97kg/m^3)$$

Here also the results obtained by the present FSDT are almost identical with those predicted by existing FSDT [5] and others like FSDT [13]. This statement is also firmly demonstrated in Fig. 4 and Fig. 5 in which the results obtained by the present theory, the FSDT [5] and the FSDT [13] are in excellent agreement for various thickness ratios and aspect ratios. Also see the obtained results by Kant [11], these values are different from those obtained by FSDT [5], FSDT [13] and the present FSDT, having to be equal o almost identical, this statement is also firmly demonstrated in Fig. 5. Kant [11] reproduced badly the FSDT proposed by Whitney and Pagano [5] and his obtained results are extensively used in the literature.

Finally, it can be mentioned that the present simplified theory were successfully validated through Navier's analytical solution. However, the formulation may be adapted to be verified through Levy solutions considering similar strategies as in this paper. But, it should be furthered investigated.

5. Conclusions

A simplified FSDT was proposed for dynamics analysis of laminates and sandwich plates. By making further simplifying assumptions to the existing FSDT, with the inclusion of an undetermined integral term, the number of unknowns and governing equations of the present FSDT are reduced by one, and hence, make the this theory simple and efficient to use. Verification studies show that the predictions by the present FSDT and existing FSDT for antisymmetric cross-ply laminate are close to each other. In conclusion, the present theory can improve the numerical computational cost due to their reduced degrees of freedom.

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Appendix A: Definition of Constants in Equation (15)

The following proposed simple technique to calculate the 'K' and 'M' element matrices (which comes from the governing Equations (11a-d) and (15)) is perhaps more convenient and simple than the others.

For K_{ij}

Calculation of N_i and M_i:

$$\begin{bmatrix} (N_1^c, M_1^c) \\ (N_2^c, M_2^c) \\ (N_6^c, M_6^c) \\ (N_4^c, M_4^c) \\ (N_5^c, M_5^c) \end{bmatrix} = (A_{ij}, B_{ij}) \begin{bmatrix} -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \alpha \\ 0 & 0 & \beta & \beta \end{bmatrix} + (B_{ij}, E_{ij}) \begin{bmatrix} 0 & 0 & 0 & -\alpha^2 \\ 0 & 0 & 0 & -\beta^2 \\ 0 & 0 & 0 & 2\alpha\beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{(A1)}$$

Where i, j = 1, 2, 3, 4, 5

First derivative of N and M with respect to x:

$$\begin{bmatrix} \frac{\partial (N_1^c, M_1^c)}{\partial x} \\ \frac{\partial (N_2^c, M_2^c)}{\partial x} \\ \frac{\partial (N_6^c, M_6^c)}{\partial x} \\ \frac{\partial (N_4^c, M_4^c)}{\partial x} \\ \frac{\partial (N_5^c, M_5^c)}{\partial x} \end{bmatrix} = (A_{ij}, B_{ij}) \begin{bmatrix} -\alpha^2 & 0 & 0 & 0 \\ 0 & -\alpha\beta & 0 & 0 \\ -\alpha\beta & -\alpha^2 & 0 & 0 \\ 0 & 0 & -\alpha^2 & -\alpha^2 \\ 0 & 0 & \alpha\beta & \alpha\beta \end{bmatrix} + (B_{ij}, E_{ij}) \begin{bmatrix} 0 & 0 & 0 & -\alpha^3 \\ 0 & 0 & 0 & -\alpha\beta^2 \\ 0 & 0 & 0 & -2\alpha^2\beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{(A2)}$$

First derivative of N and M with respect to y:

$$\begin{bmatrix} \frac{\partial(N_1^c, M_1^c)}{\partial y} \\ \frac{\partial(N_2^c, M_2^c)}{\partial y} \\ \frac{\partial(N_6^c, M_6^c)}{\partial y} \\ \frac{\partial(N_4^c, M_4^c)}{\partial y} \\ \frac{\partial(N_5^c, M_5^c)}{\partial y} \end{bmatrix} = (A_{ij},B_{ij}) \begin{bmatrix} -\alpha\beta & 0 & 0 & 0 \\ 0 & -\beta^2 & 0 & 0 \\ -\beta^2 & -\alpha\beta & 0 & 0 \\ 0 & 0 & \alpha\beta & \alpha\beta \\ 0 & 0 & -\beta^2 & -\beta^2 \end{bmatrix} + (B_{ij},E_{ij}) \begin{bmatrix} 0 & 0 & 0 & -\alpha^2\beta \\ 0 & 0 & 0 & -\beta^3 \\ 0 & 0 & 0 & -2\alpha\beta^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{A3}$$

Second partial derivative of N and M with respect to x and y:

$$\begin{bmatrix} \frac{\partial(N_1^c, M_1^c)}{\partial x \partial y} \\ \frac{\partial(N_2^c, M_2^c)}{\partial x \partial y} \\ \frac{\partial(N_6^c, M_6^c)}{\partial x \partial y} \\ \frac{\partial(N_4^c, M_4^c)}{\partial x \partial y} \\ \frac{\partial(N_5^c, M_5^c)}{\partial x \partial y} \end{bmatrix} = (A_{ij},B_{ij}) \begin{bmatrix} -\alpha^2\beta & 0 & 0 & 0 \\ 0 & -\alpha\beta^2 & 0 & 0 \\ \alpha\beta^2 & \alpha^2\beta & 0 & 0 \\ 0 & 0 & -\alpha^2\beta & -\alpha^2\beta \\ 0 & 0 & -\alpha\beta^2 & -\alpha\beta^2 \end{bmatrix} + (B_{ij},E_{ij}) \begin{bmatrix} 0 & 0 & 0 & -\alpha^3\beta \\ 0 & 0 & 0 & -\alpha\beta^3 \\ 0 & 0 & 0 & 2\alpha^2\beta^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{A4}$$

Example to get K(1,j), in Equation (11a):

From the Equations A1 and A2, $\frac{\partial N_1^c}{\partial x}$ and $\frac{\partial N_6^c}{\partial y}$ can be easily obtained and substituted in Equation A7.

$$K(1,j) = \frac{\partial N_1^c}{\partial x} + \frac{\partial N_6^c}{\partial y}, \text{ where } j=1,2,\ldots,5. \tag{A5}$$

For M_{ij}

$$\begin{bmatrix} -I_1 & 0 & 0 & -\frac{I_2}{\alpha}k_1 \\ 0 & -I_1 & 0 & -\frac{I_2}{\beta}k_2 \\ 0 & 0 & -I_1 & 0 \\ -\frac{I_2}{\alpha}k_1 & -\frac{I_2}{\beta}k_2 & 0 & -\frac{I_3}{\alpha^2}k_1^2 - \frac{I_3}{\beta^2}k_2^2 \end{bmatrix} \tag{A6}$$

Following the same technique the coefficients associated with the rest of the governing equations can be obtained, and in this way the system of equations, see Eq. (15), can be solved.

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Table Headings

Table 1.

Displacement models.

Table 2.

Dimensionless fundamental frequency of antisymmetric cross-ply $(0/90)_n$ square laminates (Material 1, $a = 5h$).

Table 3.

Dimensionless fundamental frequency of antisymmetric cross-ply $(0/90)_n$ square laminates (Material 1, $E_1/E_2 = 40$).

Table 4.

Dimensionless fundamental frequency of antisymmetric $(0/90/core/0/90)$ sandwich plate laminates (Material 2, $a/b = 1, t_c/t_f = 10$).

Table 5.

Dimensionless fundamental frequency of antisymmetric $(0/90/core/0/90)$ sandwich plate laminates (Material 2, $a/b = 1, a/h = 10$).

Table 6.

Dimensionless fundamental frequency of antisymmetric $(0/90/core/0/90)$ sandwich plate laminates (Material 2, $t_c/t_f = 10, a/h = 10$).

Figure Legends

Figure 1. Coordinate system and layer numbering used for a typical laminate.

Figure 2. Variation of dimensionless fundamental frequency of antisymmetric cross-ply $(0/90)_n$ square laminates versus material anisotropic (Material 1, $a = 5h$).

Figure 3. Variation of dimensionless fundamental frequency of antisymmetric cross-ply $(0/90)_n$ square laminates versus thickness ratio (Material 1, $E_1/E_2 = 40$).

Figure 4. Variation of dimensionless fundamental frequency of antisymmetric $(0/90/core/0/90)$ sandwich plate laminates versus thickness ratio (Material 2, $a/b = 1$, $t_c/t_f = 10$).

Figure 5. Variation of dimensionless fundamental frequency of antisymmetric $(0/90/core/0/90)$ sandwich plate laminates versus aspect ratio (Material 2, $t_c/t_f = 10$, $a/h = 10$).

Tables

Table 1.

Model	Theory	Unknowns
CPT	Classical plate theory	3
FSDT	First-order shear deformation theory (Whitney and Pagano [5])	5
FSDT	First-order shear deformation theory (Thai [13])	4
TSDT	Third-order shear deformation theory (Reddy [34])	5
RPT1	Refined plate theory 1 (Thai [12])	4

RPT2	Refined plate theory 2 (Thai [12])	5
Present	New FSDT	4

Table 2.

E1/E2	Theory	n			
		1	2	3	5
3	Exact [40]	6.2578	6.5455	6.6100	6.6458
	TSDT [12]	6.2169	6.5008	6.5558	6.5842
	FSDT [11]	6.1490	6.4402	6.4916	6.5185
	FSDT [12]	6.2085	6.5043	6.5569	6.5837
	FSDT [13]	6.2085	6.5043	6.5569	6.5837
	Present	6.2085	6.5043	6.5569	6.5837
10	Exact [40]	6.9845	8.1445	8.4143	8.5625
	TSDT [12]	6.9887	8.1954	8.4052	8.5126
	FSDT [11]	6.9156	8.1963	8.3883	8.4842
	FSDT [12]	6.9392	8.2246	8.4183	8.5132
	FSDT [13]	6.9392	8.2246	8.4183	8.5132
	Present	6.9392	8.2246	8.4183	8.5132
20	Exact [40]	7.6745	9.4055	9.8398	10.0843
	TSDT [12]	7.8210	9.6265	9.9181	10.0674
	FSDT [11]	7.6922	9.6729	9.9266	10.0483
	FSDT [12]	7.7060	9.6885	9.9427	10.0638
	FSDT [13]	7.7060	9.6885	9.9427	10.0638
	Present	7.7060	9.6885	9.9427	10.0638
30	Exact [40]	8.1763	10.1650	10.6958	11.0027
	TSDT [12]	8.5050	10.5348	10.8547	11.0197
	FSDT [11]	8.3112	10.6095	10.8723	10.9959
	FSDT [12]	8.3211	10.6198	10.8828	11.0058
	FSDT [13]	8.3211	10.6198	10.8828	11.0058
	Present	8.3211	10.6198	10.8828	11.0058
40	Exact [40]	8.5625	10.6789	11.2728	11.6245
	TSDT [12]	9.0871	11.1716	11.5012	11.6730
	FSDT [11]	8.8255	11.2635	11.5189	11.6374
	FSDT [12]	8.8333	11.2708	11.5264	11.6444
	FSDT [13]	8.8333	11.2708	11.5264	11.6444
	Present	8.8333	11.2708	11.5264	11.6444

Note-1: Thai [12] reproduced the FSDT proposed by Whitney and Pagano [5].

Note-2: Kant [11] reproduced the FSDT proposed by Whitney and Pagano [5].

Table 3.

a/h	Theory	n			
		1	2	3	5
2	TSDT [34]	5.7170	5.7546	5.8741	5.9524
	FSDT [12]	5.2104	5.6656	5.6992	5.7140
	Present	5.2104	5.6656	5.6992	5.7140
	RPT1	5.7170	5.7546	5.8741	5.9524
	RPT2	5.5017	5.7240	5.8180	5.8721
	CLPT	8.6067	14.1036	15.0895	15.6064
4	TSDT [34]	8.3546	9.7357	9.9878	10.1241
	FSDT [12]	8.0349	9.8148	9.9852	10.0628
	Present	8.0349	9.8148	9.9852	10.0628
	RPT1	8.3546	9.7357	9.9878	10.1241
	RPT2	8.2651	9.7357	9.9855	10.1167
	CLPT	10.4244	16.3395	17.2676	17.7314
10	TSDT [34]	10.5680	14.8463	15.4632	15.7700
	FSDT [12]	10.4731	14.9214	15.5010	15.7790
	Present	10.4731	14.9214	15.5010	15.7790
	RPT1	10.5680	14.8463	15.4632	15.7700
	RPT2	10.5480	14.8433	15.4627	15.7700
	CLPT	11.1537	17.1448	18.0461	18.4916
20	TSDT [34]	11.1052	16.5733	17.3772	17.7743
	FSDT [12]	11.0779	16.6008	17.3926	17.7800
	Present	11.0779	16.6008	17.3926	17.7800
	RPT1	11.1052	16.5733	17.3772	17.7743
	RPT2	11.0997	16.5719	17.3769	17.7743
	CLPT	11.2693	17.2682	18.1652	18.6080
50	TSDT [34]	11.2751	17.1849	18.0644	18.4984
	FSDT [12]	11.2705	17.1899	18.0673	18.4995
	Present	11.2705	17.1899	18.0673	18.4995
	RPT1	11.2751	17.1849	18.0644	18.4984
	RPT2	11.2742	17.1847	18.0643	18.4984
	CLPT	11.3023	17.3032	18.1990	18.6410
100	TSDT [34]	11.3002	17.2784	18.1698	18.6097
	FSDT [12]	11.2990	17.2796	18.1706	18.6100
	Present	11.2990	17.2796	18.1706	18.6100

RPT1	11.3002	17.2784	18.1698	18.6097
RPT2	11.2999	17.2783	18.1698	18.6097
CLPT	11.3070	17.3082	18.2038	18.6457

Note-3: Thai [12] reproduced the FSDT proposed by Whitney and Pagano [5].

Table 4.

a/h	Theories			
	FSDT [11]	FSDT [5]	FSDT [13]	Present
2	5.2017	5.6114	5.6114	5.6114
4	9.0312	9.5447	9.5447	9.5447
10	13.8694	14.1454	14.1454	14.1454
20	15.5295	15.6124	15.6124	15.6124
30	15.9155	15.9438	15.9438	15.9438
40	16.0577	16.0655	16.0655	16.0655
50	16.1264	16.1229	16.1229	16.1229
60	16.1612	16.1544	16.1544	16.1544
70	16.1845	16.1735	16.1735	16.1735
80	16.1991	16.1859	16.1859	16.1859
90	16.2077	16.1944	16.1944	16.1944
100	16.2175	16.2006	16.2006	16.2006

Note - 4: Results using these theories [5] and [13] are computed independently by the authors.

Table 5.

tc/tf	Theories			
	FSDT [11]	FSDT [5]	FSDT [13]	Present
4	13,9190	13,3307	13,3307	13,3307
10	13,8694	14,1454	14,1454	14,1454
20	12,8946	13,9939	13,9939	13,9939
30	11,9760	13,5209	13,5209	13,5209
40	11,2036	13,0152	13,0152	13,0152
50	10,5557	12,5338	12,5338	12,5338

100	8,4349	10,6571	10,6571	10,6571

Note - 5: Results using these theories [5] and [13] are computed independently by authors.

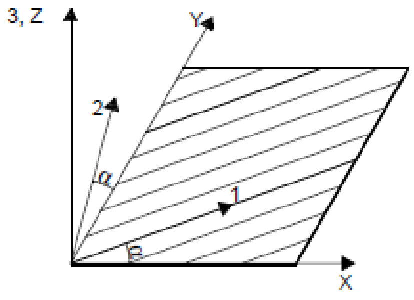
Table 6.

a/b	Theories			
	FSDT [11]	FSDT [5]	FSDT [13]	Present
0,5	39,4840	40,1511	40,3559	40,3559
1	13,8694	14,1454	14,1454	14,1454
1,5	9,4910	9,7826	9,8376	9,8376
2	10,1655	7,9863	8,0759	8,0759
2,5	6,5059	6,8463	6,9340	6,9340
3	5,6588	5,9993	6,0727	6,0727
5	3,6841	3,9658	3,9929	3,9929

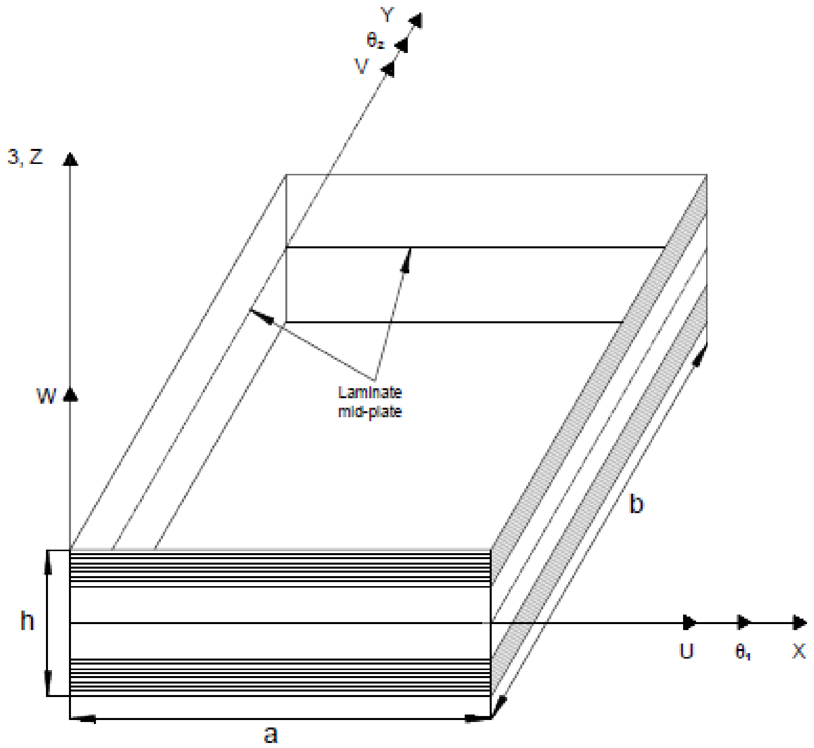
Note - 6: Results using these theories [5] and [13] are computed independently by the authors.

Figures

Fig. 1

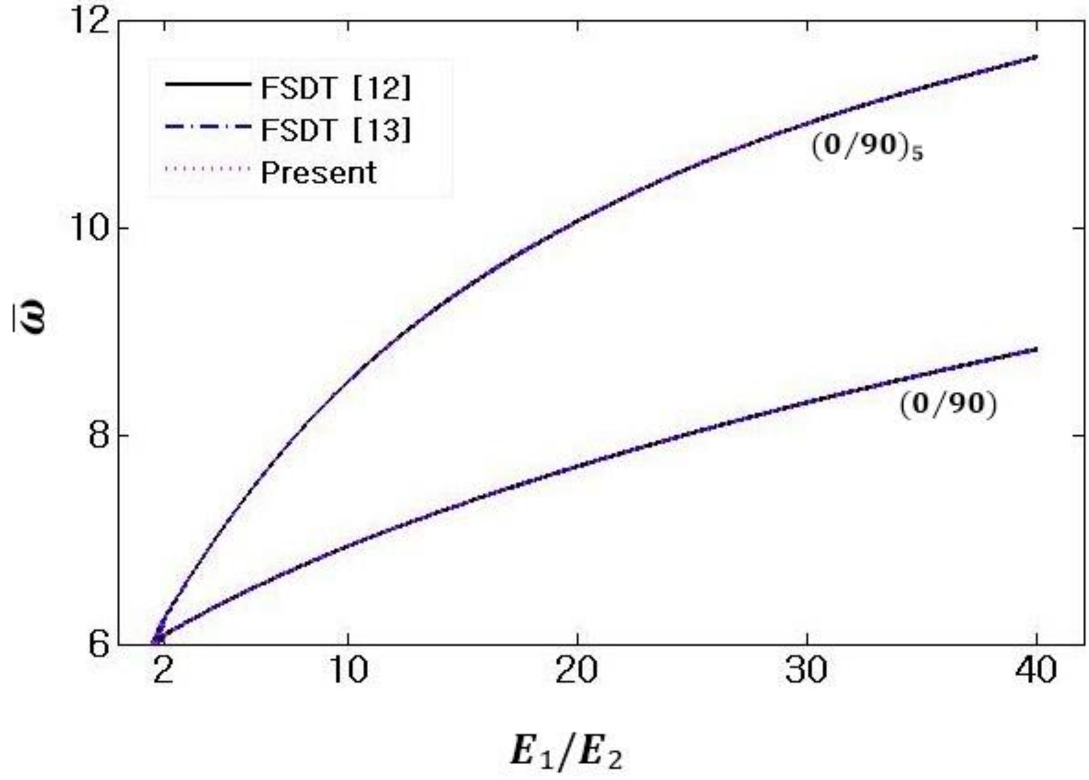


(1,2,3) - Lamina reference axes



(X,Y,Z) - Lamina reference axes

Fig. 2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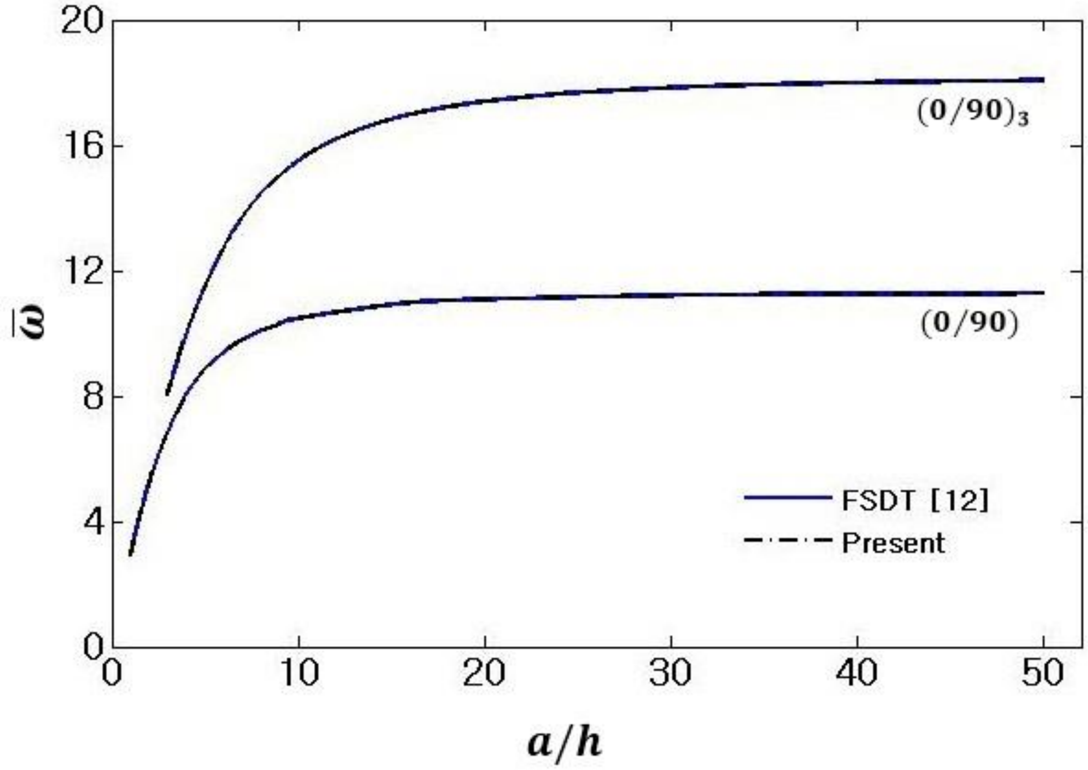
FSDT [13]
Present
$(0/90)_5$
$(0/90)$
$\bar{\omega}$
E_1/E_2

Fig. 3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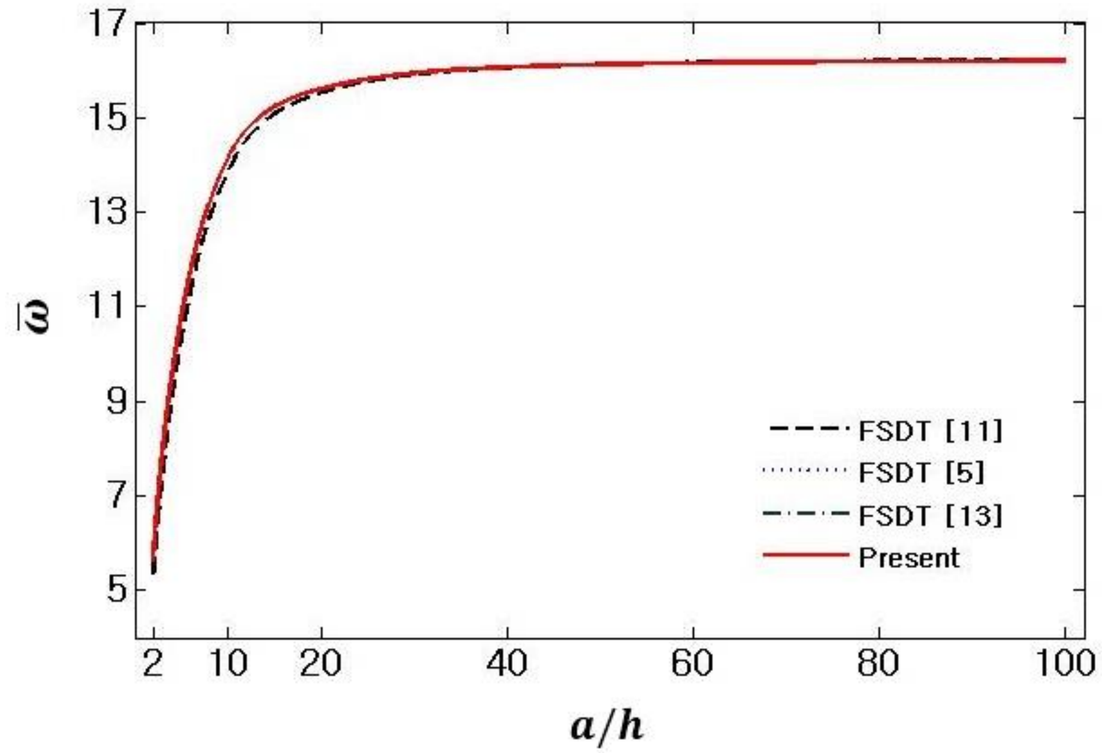
$(0/90)$
FSDT [12]
Present
$\bar{\omega}$
a/h

Fig 4.



17
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$\bar{\omega}$
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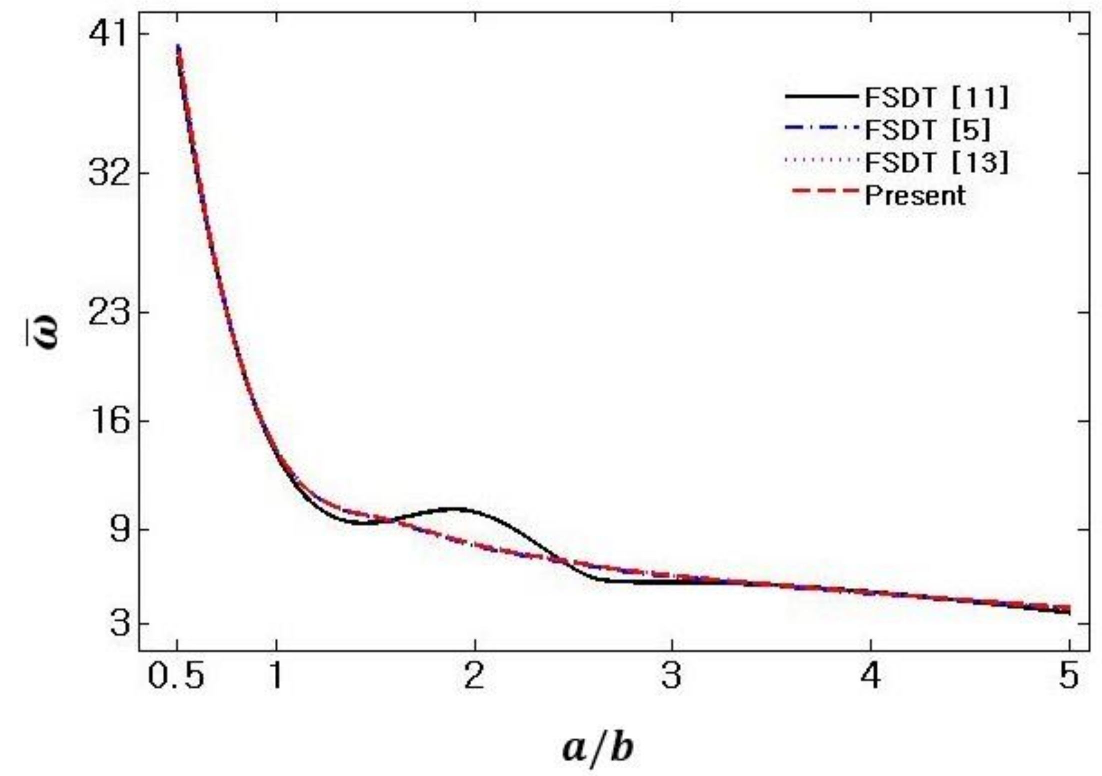
100
a/h
FSDT [11]
FSDT [5]
FSDT [13]
Present

Fig 5.



41
32
23
16
9
3
$\bar{\omega}$
0.5
1
2
3
4
5
a/b
FSDT [11]
FSDT [5]
FSDT [13]
Present